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APPROXIMATE CALCULATION METHOD FOR ONE-DIMENSIONAL GAS FLOWS INDUCED BY NONMONOTONIC MOTION OF A PISTON

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The problem of one-dimensional piston which at the beginning moves with increasing velocity into a gas at rest, then is decelerated, and finally stops, is solved by means of special series. The gas flow field is constructed by a successive joining of three characteristic Cauchy problems in terms of their characteristic solutions. Generalized solution of the problem of instantaneous arrest of the piston is derived. Obtained equations are used for the approximate calculation of the motion of generated shock waves.

Representation of solutions of certain boundary value problems for nonlinear equations of the hyperbolic kind in the form of special series was proposed in [1, 2]. The problem of the piston moving into a gas at rest is solved there, and the obtained solution was used for an approximate determination of the generated shock wave. The piston velocity was assumed to be monotonically increasing. That problem is solved here with the use of similar series in the case when the piston velocity is nonmonotonous,

Numerical methods make it possible at present to determine one-dimensional flows similar to that considered below, and multidimensional problems can be solved by the method proposed in [1, 2]. The use of the proposed scheme for solving the problem of the multidimensional piston, whose velocity is nonmonotonous, does not present theoretical difficulties, but except that the formulas are more cumbersome.

1. One-dimensional isentropic flows of perfect polytropic gas are considered. The potential of these satisfies the equation

$$\Phi_{tt} + 2\Phi_{tx}\Phi_x + (\Phi_x^2 - c^2)\Phi_{xx} - (i-1)c^2\frac{\Phi_x}{x} = 0, \quad x > 0$$

$$c^2 = (\gamma - 1)(M - \Phi_t - {}^1/_2\,\Phi_x^2), \quad M = 1 / (\gamma - 1)$$

where c is the speed of sound, $\gamma > 1$ is the adiabatic exponent, and $i = 1$, $i = 2$ and $i = 3$ relate, respectively, to plane, cylindrical, and spherical symmetry.

Let an impermeable piston begin to penetrate at instant of time $t = 0$ into a homogeneous quiescent gas in conformity with the law $x = \chi_1(t)$, and

$$\chi_1(0) = R_0 > 0, \quad \chi_1'(0) = 0, \quad \chi_1''(t) > 0, \quad 0 \leqslant t \leqslant t_1 \tag{1.1}$$

A surface of weak discontinuity begins to propagate through the quiescent gas. This discontinuity is a characteristic which separates the region of the quiescent gas from that of its perturbed flow. A characteristic Cauchy problem (problem 1) is obtained: along the characteristic are specified the initial conditions: the homogeneous gas is at rest, and along the line $x = \chi_1(t)$ the boundary condition

$$\Phi_x(\chi_1(t),\ t) = \chi_1'(t) \tag{1.2}$$

which is the corollary of the piston impermeability.

From the instant t_1 the piston begins to penetrate the gas in conformity with the law $x = \chi_2(t)$, and

$$\chi_2(t_1) = \chi_1(t_1),\quad \chi_2'(t_1) = \chi_1'(t_1)$$

$$\chi_2''(t) < 0,\quad t_1 \leqslant t \leqslant t_2;\quad \chi_2'(t_2) = 0$$

One more weak discontinuity surface is generated in the perturbed flow, which is completely determined by the solution of problem 1. The specified flow of gas on one side of that characteristic and the boundary condition at the piston, similar to condition (1.2), also constitute a characteristic Cauchy problem (problem 2).

At the instant of time $t = t_2$ the piston stops, hence $x = \chi_3(t) = \text{const}$. One more weak discontinuity determined by the solution of problem 2, is generated. By specifying on the latter the initial conditions from the solution of problem 2 and on the piston conditions of the kind (1.2) we obtain one more characteristic Cauchy problem (problem 3).

2. For the successive solution of the three problems formulated above we pass, as in [1, 2], to new independent variables r and t, and to the new unknown function Ψ by the Legendre transformation $\Psi = rx + Mt - \Phi$ whose Jacobian is $J = \Psi_r\Psi_{rr}$. As the result, we obtain for Ψ the equation

$$\Psi_{tt}\Psi_{rr} - \Psi_{rt}^2 + 2r\Psi_{rt} - r^2 + c^2 + (i-1)rc^2\frac{\Psi_{rr}}{\Psi_r} = 0 \tag{2.1}$$

$$c^2 = (\gamma - 1)(\Psi_t - {}^1/_2\, r^2), \tag{2.2}$$

The corollary of Legendre transformation are the formulas

$$x = \Psi_r,\quad r = \Phi_x,\quad \Psi_{rr} = 1 / \Phi_{xx} \tag{2.3}$$

The characteristic Cauchy problem for Eq. (2.1) is generally formulated as follows. Let in some region Γ of the plane tOr the solution of Eq. (2.1) be specified by

$$\Psi(r, t) = f(r, t) \tag{2.4}$$

and let point $(t = t_0, r = r_0)$ belong to Γ. Then the characteristic $r - \varphi(t) = 0$ solution (2.4) which passes through the specified point satisfies the equation and the initial condition

$$[f_{tt}(\varphi, t) + (i-1)\varphi(\gamma-1)(f_t(\varphi, t) - {}^1/_2\,\varphi^2) / f_r(\varphi, t)] - \\ [2\varphi - 2f_{rt}(\varphi, t)]\varphi' + [f_{rr}(\varphi, t)]\varphi'^{2} = 0,\quad \varphi(t_0) = r_0 \tag{2.5}$$

Let us assume that problem (2.5) has a solution. If there are two of these, we select the one that conforms to the physical meaning of the problem.

We introduce new independent variables $z = r - \varphi(t)$ and $\tau = t$ which means that the characteristic is taken as one of the coordinates. Equation (2.1) now becomes

$$(\Psi_{\tau\tau} - 2\varphi'\Psi_{z\tau} + \varphi'^2\Psi_{zz} - \varphi''\Psi_z)\Psi_{zz} - (\Psi_{z\tau} - \\ \varphi'\Psi_{zz})^2 + 2(z+\varphi)(\Psi_{z\tau} - \varphi'\Psi_{zz}) - (z+\varphi)^2 + (\gamma-1)\times \\ (\Psi_\tau - \varphi'\Psi_z - {}^1/_2(z+\varphi)^2) + (i-1)(\gamma-1)(z+\varphi)(\Psi_\tau - \\ \varphi'\Psi_z - {}^1/_2(z+\varphi)^2)\Psi_{zz} / \Psi_z = 0 \tag{2.6}$$

for which the initial conditions are

$$\Psi(0, \tau) = f(\varphi(\tau), \tau), \quad \Psi_z(0, \tau) = f_r(\varphi(\tau), \tau) \tag{2.7}$$

The boundary condition for the characteristic Cauchy problem (2.6),(2.7) is obtained, as in [3], from the relationship

$$\chi(\tau) = \Psi_z(z, \tau)\,|_{z=\chi'(\tau)-\varphi(\tau)} \tag{2.8}$$

Equation (2.8) follows from formulas (2.3) and the condition of impermeability of the piston moving in conformity with the law $x = \chi(t)$. Differentiation of (2.8) with respect to τ and the introduction of function η inverse of $\chi' - \varphi$ yield for the characteristic Cauchy problem (2.6),(2.7) the boundary condition

$$\Psi_{zz}\,|_{\tau=\eta(z)} = \eta'(z)\,[\chi', \eta(z)) - \Psi_{z\tau}]\,|_{\tau=\eta(z)} \tag{2.9}$$

To satisfy Eq. (2.9) in the neighborhood of point $(\tau = t_0,\ z = 0)$ it is necessary to specify the condition

$$\chi''(t_0) - \varphi'(t_0) \neq 0 \tag{2.10}$$

The solution of the characteristic Cauchy problem (2.6),(2.7),(2.9) is sought in the form of series in powers of z

$$\Psi(z, \tau) = \sum_{k=0}^{\infty} \frac{T_k(\tau)}{k!} z^k = \sum_{k=0}^{\infty} \frac{T_k(t)}{k!} (r - \varphi(t))^k \tag{2.11}$$

Then T_0 and T_1 are determined by (2.7), and T_2 and T_k $(k \geqslant 3)$ satisfy the equations

$$BT_2' + CT_2^2 + DT_2 + E = 0 \tag{2.12}$$
$$BT_k' + D_kT_k + E_k = 0, \quad k = 3, 4, \ldots$$
$$B = -2\,(T_1'(\tau) - \varphi(\tau))$$

where C, D and E are known functions of τ when $T_2, \ldots, T_{k-1}$ are known. If in the space of variables x and t there are no points on the characteristic $x = \xi(t)$, which corresponds to characteristic $z = 0$, at which the speed of sound is zero, then $B \neq 0$ since it follows from formulas (2.3) that

$$\frac{d}{d\tau}[f_r(\varphi(\tau), \tau)] = \frac{d}{d\tau}[\xi(\tau)], \quad \varphi(\tau) = \Phi_x(\xi(\tau), \tau)$$

For fairly small t we have in the piston problem $B \neq 0$ in all three characteristic Cauchy problems.

Initial conditions for Eqs. (2.12) are obtained from formula (2.9). The convergence of series (2.11) in some neighborhood of points $(\tau = t_0,\ z = 0)$ follows from the theorem proved in [3].

3. Solutions of the characteristic Cauchy problems specified in Sect. 1 are as follows: In problem 1 [1, 2]

$$z = r, \ \Psi(0, t) = K + t / (\gamma - 1), \ \Psi_r(0, t) = t + R_0, \quad K = \text{const} \tag{3.1}$$

Formulas (1.1) ensure that conditions (2.10) are satisfied in problem 1.

For $i = 3$ the solution was derived in [2], for $i = 2$ it appears in [1], and for $i = 1$ the solution of problem 1 is of the form

$$\Psi_1(r, t) = \left(K + \frac{t}{\gamma - 1}\right) + (t + R_0)r + \left(\frac{\gamma+1}{4}t + C_2\right)r^2 + \sum_{k=3}^{\infty} C_k r^k \quad (3.2)$$

$C_k = \text{const}, \quad k \geqslant 2$

To solve problem 2 it is necessary to construct the characteristic $r = \varphi_1(t)$ beginning at point ($t = t_1$, $r = \chi_1'(t_1)$) using the solution of problem 1. For fairly small t_1 and $\chi_1'(t_1)$ Eq. (2.5) has two solutions: $\varphi_{11}'(t_1) > 0$ and $\varphi_{12}'(t_1) \leqslant 0$. Since for $t \geqslant t_1$ the piston is decelerated, we select for $\varphi_1(t)$ the second of these solutions.

For $i = 1$ we have $\varphi_1(t) = \chi_1'(t_1)$. For $i = 2$ and $i = 3$ Eq. (2.5) was solved numerically, using finite segments of infinite series

$$\sum_{k=0}^{3} \frac{T_k(\tau)}{k!} z^k \quad (3.3)$$

Having constructed $\varphi_1(t)$ we select $\chi_2(t)$ so that the condition similar to (2.10) is satisfied for $t = t_1$.

For $i = 1$ the solution of problem 2 is as follows:

$$\Psi_2(r, t) = \Psi_1(r_1, t) + \Psi_{1r}(r_1, t)(r - r_1) + \left[\frac{\gamma+1}{4}t + C_{22}\right] \times \quad (3.4)$$

$$(r - r_1)^2 + \sum_{k=3}^{\infty} C_{k2}(r - r_1)^k$$

$r_1 = \chi_1'(t_1), \; C_{k2} = \text{const}, \; k \geqslant 2$

For solving problem 2 for $i = 2$ and $i = 3$ finite segments of (3.3) were taken instead of solutions of problem 1, and Eqs. (2.12) were solved numerically.

To obtain the solution of problem 3 it is necessary to construct the characteristic $r = \varphi_2(t)$ for the solution of problem 2, beginning at point ($t = t_2$, $r = 0$). For fairly small t_2 and $\chi_1'(t_1)$ Eq. (2.5) has two solutions: $\varphi_{21}'(t_2) < \varphi_{22}'(t_2) \leqslant 0$. We select $\varphi_{22}(t)$ for $\varphi_2(t)$, since the selection of $\varphi_{21}(t)$ would result in that in the flow restored to the physical plane the characteristic "propagates" with increasing time into the piston instead of into the gas.

For $i = 1$ we have $\varphi_2(t) = 0$, and, since $\chi_3(t) = \text{const}$, condition (2.10) is not satisfied, and it is not possible to construct a solution of problem 3 in the space of variables r, t. If problem 3 is analyzed in the space of variables x, t, then it follows from (3.4), (2.2) and (2.3) that the characteristic which separates the solutions of problems 2 and 3 is a straight line. Along that line $U = 0$ and $\rho = \rho_0 = \text{const}$, where ρ_0 is the density of gas at $t = 0$. Thus problem 3 is defined by

$$\rho_t + U\rho_x + \rho U_x = 0, \quad \rho U_t + \rho U U_x + \rho^{\gamma-1}\rho_x = 0 \quad (3.5)$$

$$U = 0, \; \rho = \rho_0 \quad \text{on the line } x = t - t_2 + \chi_2(t_2)$$

$$U = 0 \quad \text{for } x = \chi_2(t_2)$$

It can be shown that problem (3.5) has the single analytic solution

$$U = 0, \; \rho = \rho_0 \quad (3.6)$$

Thus, (3.2), (3.4) and (3.6) solve the input problem of the piston. Solutions (3.2) and (3.4) represent simple waves [4].

For $i = 2$ and $i = 3$ we have $\varphi_2'(t_2) < 0$. Equation (2.5) was solved numerically. Its finite segment (3.3), determined by the segment (3.3) of the solution of problem 1, was substituted for the solution of problem 2.

The results of computations carried out for

$$\chi_1(t) = 1 + {}^1/_2 t^2 - {}^4/_6 t^3, \quad t_1 = {}^1/_4$$

$$\chi_2(t) = \chi_1(t_1) + \chi_1'(t_1)(t - t_1) - {}^1/_8 (t - t_1)^2, \quad t_2 = {}^3/_4$$

are shown in Figs. 1 — 3, where lines denoted by small circles, black dots, and those without these relate to $i = 1$, $i = 2$ and $i = 3$, respectively.

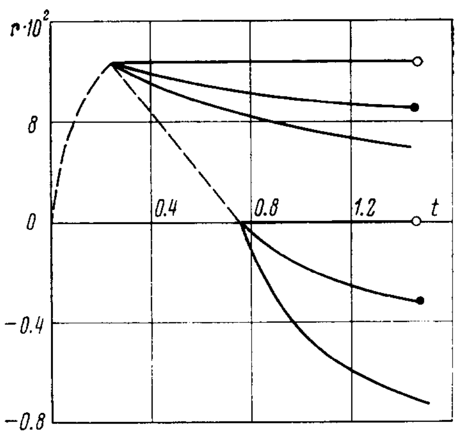

Fig. 1

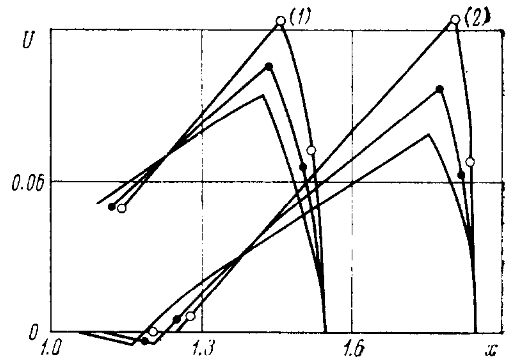

Fig. 2

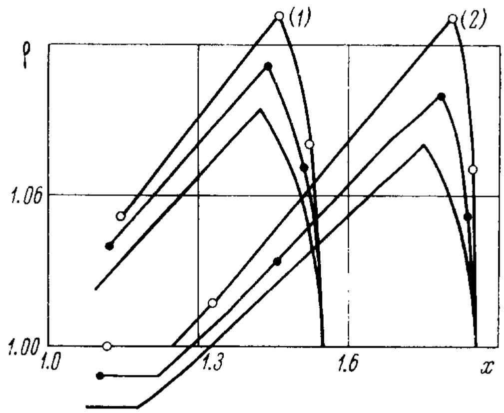

Fig. 3

Functions $\varphi_1(t)$ and $\varphi_2(t)$ are represented in Fig. 1 by the upper and lower bunches of solid lines, respectively, and the dash lines relate to $\chi_1'(t)$ and $\chi_2'(t)$.

Curves of gas velocity U and density ρ determined by (3.3) of the solution of problem 3 are shown in Figs. 2 and 3 for instances of time $t = 0.55$ (curves 1) and $t = 0.85$ (curves 2).

4. The shortening of the deceleration time, i.e. when $t_2 \to t_1$ yields "at the limit" the problem of instantaneous piston arrest. Solution of the "limit" problem in the space of variables x and t is difficult, hence the transition to limit $t_2 \to t_1$ is considered in the space of variables r and t.

The boundary condition for problem 2 is of the form

$$\Psi_{rr}(r, \eta_2(r)) = \eta_2'(r)\,[r - \Psi_{rt}(r, \eta_2(r))] \tag{4.1}$$

where η_2 is an inverse function of χ_2'; $\eta_2(r_1) = t_1$, $\eta_2(0) = t_2$ and $r_1 = \chi_1'(t_1)$. When $t_2 \to t_1$, η_2 tends to merge with the straight line $t = t_1$, and condition (4.1)

becomes
$$\Psi_{rr}(r, t_1) = \Psi_{zz}(z, t_1) = 0 \tag{4.2}$$

Problem 2 with boundary condition (4.2) instead of (2.9) has no singularities and has also an analytic solution. Because of this, a singularity of the flow can only occur at points where $J = 0$, which specifically happens when $t = t_1$. Since T_2 satisfies in problem 2 the nonhomogeneous equation, there exists a $t_1' > t_1$ such that $T_2(t) \neq 0$ when $t_1 < t < t_1'$. Consequently, the flow in some neighborhood of the characteristic in the physical space has no singularities for $t_1 < t < t_1'$.

The derived general solutions for $i = 1$ represent a centered rarefaction wave, while for $i = 2$ and $i = 3$ they resemble a centered rarefaction wave.

5. With increasing time infinite gradients begin to appear in the solution of problem 1, a shock wave is generated, and its isentropic property is violated. The place and time $t = t_*$ of shock wave generation can be determined with the use of the solution of problem 1 [1, 2].

If the shock wave is weak, it can be approximately assumed that the flow behind it is isentropic and potential. The solution of problem 1 can be used for the approximate determination of the motion of the generated shock wave and for defining the flow of gas downstream of it [1, 2]. If $R(t)$ defines the flow of gas at the shock wave, the equation
$$\Psi_{rr}(R(t), t)\, R'(t) + \Psi_{rt}(R(t), t) = \frac{\gamma+1}{4} R(t) + \sqrt{\frac{(\gamma+1)^2}{16} R^2(t) + 1} \tag{5.1}$$

which is the corollary Hugoniot's condition [4], is valid. Equation (5.1) can be solved numerically by substituting segment (3.3) for Ψ.

The characteristic which separates the solutions of problems 1 and 2 catches up with the shock wave at some instant of time $t = t_{**}$. It is assumed that in that case the result of the interaction between the weak discontinuity and the shock wave is only a shock wave that propagates over the quiescent region. Then for the approximate determination of shock wave motion beginning at $t = t_{**}$ we use the solution of problem 2 and in Eq. (5.1) we substitute (3.3) of solution of problem 2 for Ψ when $t \geqslant t_{**}$.

Note. In the exact solution the interaction between a weak discontinuity and a shock wave results not only in a shock wave propagating through the region at rest but, also, in a weak discontinuity penetrating the interior of the flow. However, the use of isentropic potential flows for approximately defining the motion of gas behind the shock wave leads to the following. A weak discontinuity that propagates in the interior of flows yields the characteristic Cauchy problem whose solution is nonunique and is represented by a line along which the boundary condition (which ensures uniqueness) is specified. That characteristic can only be known, if the law of motion of the shock wave is known. If the new characteristic Cauchy problem is disregarded, the shock wave motion is uniquely defined by the solution of the related differential equation. The interaction between a weak shock wave and a rarefaction wave catching up with it is approximately defined in [4] as follows: the generated weak discontinuity is disregarded, and the flow behind the shock wave is defined by the initial rarefaction wave, and the position of the shock wave is uniquely determined by the solution of the related differential equation.

Shock wave motion computed for $i = 3$, $\chi_1(t) = 1 + {}^1/_2\, t^2 - 1.28\, t^3 / 6$ and $t_1 = t_2 = {}^3/_4$ are represented in Fig. 4 (parameter $\rho - 1$ is close to U and is not adduced here).

Beyond the solution of problem 2 the gas is at rest for $i = 1$. For $i = 2$ and $i = 3$ it is not so, and there exists an instant of time at which Φ_{xx} in the solution of problem 3 becomes infinite. This may be treated as the appearance of the rear boundary of an N-wave [4]. It is not clear at present which solution is to be used for defining the flow between the rear boundary of the N-wave and the piston. Hence it is impossible to state anything about the position of that boundary, except that it will not reach that point of the flow which is defined by the solution of problem 2, where $U = 0$.

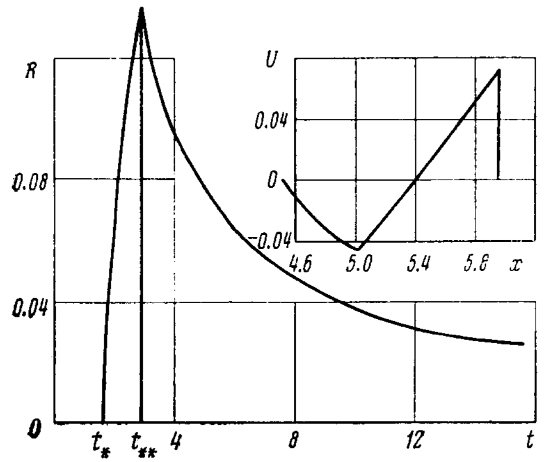

Fig. 4

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